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ON THE SOLUTIONS OF ELASTIC–PLASTIC PROBLEMS WITH CONTACT-TYPE BOUNDARY CONDITIONS FOR SOLIDS WITH LOSS-OF-STRENGTH ZONES†

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A formulation of a quasi-static problem of the mechanics of elastic–plastic bodies with loss-of-strength zones and boundary conditions of contact type is given which enables the properties of the loading system to be taken into account. With certain constraints on the constitutive relations and using a condition for stability of the softening process in a local zone, theorems are proved on the uniqueness of the solution of the boundary-value problem and on the maximum and minimum of the functionals when the kinematically or statically possible and actual fields are the same. The corresponding generalized variational principles are given. © 1998 Elsevier Science Ltd. All rights reserved.

1. In an arbitrary Cartesian system of coordinates, let the constitutive relations which associate increments of the stress tensor $d\sigma$ and the strain tensor $d\varepsilon$ under continuous loading of an element of the material be given in linear tensor form

$$d\sigma_{ij} = C_{ijmn}(\varepsilon, \chi)d\varepsilon_{mn} \tag{1.1}$$

where the parameter χ is equal to unity during active loading, when $\sigma_{ij}d\varepsilon_{ij} > 0$, and zero when the load is removed. In the latter case, the behaviour of the material is governed by the constant modulus-of-elasticity tensor $\mathbf{C}^e$. We shall confine our analysis to materials which possess soft characteristics [1], for which

$$\mathbf{C}(\varepsilon, \chi = 1) \leqslant \mathbf{C}(\varepsilon = 0, \chi) = \mathbf{C}^e$$

It is assumed that there is an initial stress–strain state, so that at an instant of time preceding the time considered there is a known non-zero stress field $\sigma(\mathbf{x})$, strain field $\varepsilon(\mathbf{x})$ and displacement field $\mathbf{u}(\mathbf{x})$. The total strains, and their increments, have elastic and plastic components

$$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p, \quad d\varepsilon_{ij} = d\varepsilon_{ij}^e + d\varepsilon_{ij}^p$$

We will assume the existence of limiting surfaces, i.e. of a loading surface in stress space and a strain surface in strain space. By the Mises maximum principle of the rate of dissipation [2]

$$(\sigma_{ij} - \sigma_{ij}^*)d\varepsilon_{ij}^p \geqslant 0 \tag{1.2}$$

where σ_{ij} are the real values of the components of the stress tensor, corresponding to the limiting surface for a given value of ε_{ij}^p and σ_{ij}^* are the components of any possible stress state which is allowed by the given load function. It follows from this inequality that the loading and strain surfaces are non-concave, and the vector of the plastic deformation increment is along the outward normal to the limiting surface.

The strain increments are small, and thus Cauchy's relations, which connect them and the displacement increment vector, hold, namely

$$d\varepsilon_{ij} = \frac{1}{2}\left[\frac{\partial}{\partial x_j}(du_i) + \frac{\partial}{\partial x_i}(du_j)\right] \tag{1.3}$$

and the equations of the medium equilibrium are satisfied ($\mathbf{X}$ are given volume forces)

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$$d\sigma_{ij,j} + DX_i = 0 \tag{1.4}$$

The loading conditions of a body Ω with boundary $\Sigma = \Sigma_S + \Sigma_u$ are defined using boundary conditions of the contact type [3] in the form

$$(d\sigma_{ij}n_j + R_{ij}du_j)\,|_{\Sigma_S} = d\overset{\circ}{S_i}, \quad (du_i + Q_{ij}d\sigma_{jk}n_k)\,|_{\Sigma_u} = d\overset{\circ}{u_i} \tag{1.5}$$

This enables additional information to be provided on the stiffness characteristics $R_{ij}(\mathbf{u}, \mathbf{x})$ and the compliance characteristics $Q_{ij}(\mathbf{S}, \mathbf{x})$ of the loading system [4], which satisfy the conditions

$$\forall \mathbf{e} \quad R_{ij}e_je_i \geqslant 0, \quad Q_{ij}e_je_i \geqslant 0, \quad R_{ik}Q_{kj} = \delta_{ij} \tag{1.6}$$

Here n_k are the direction cosines of the vector normal to the area and δ_{ij} is the Kronecker delta. Nominally, ignoring the strain or resistance of the body, the given increments of the forces and displacements on the boundary are related by the equations

$$dS_i^\circ = R_{ij}du_j^\circ, \quad du_i^\circ = Q_{ij}dS_j^\circ \tag{1.7}$$

and from (1.6) we obtain the relation of Eqs (1.5). In the general case this enables us to use boundary conditions of the same form for the entire surface. Thus for (1.3)–(1.5) we obtain the relations

$$\int\limits_{\Sigma=\Sigma_S} dS_i^\circ du_i d\Sigma = \int\limits_{\Omega} (d\sigma_{ij}d\varepsilon_{ij} - dX_i du_i)d\Omega + \int\limits_{\Sigma=\Sigma_S} R_{ij}du_j du_i d\Sigma \tag{1.8}$$

$$\int\limits_{\Sigma=\Sigma_u} dS_i du_i^\circ d\Sigma = \int\limits_{\Omega} (d\sigma_{ij}d\varepsilon_{ij} - dX_i du_i)d\Omega + \int\limits_{\Sigma=\Sigma_u} Q_{ij}dS_j dS_i d\Sigma \tag{1.9}$$

where $dS_i = d\sigma_{ij}n_j\,|_\Sigma$.

Equations (1.8) and (1.9) are similar to the equation of virtual work [5] and are taken as the basis of the proof of the fundamental theorems of the mechanics of inelastic deformation of bodies with contact-type boundary conditions. In the case where, in addition to possible loss of strength of the material, the stress level falls during progressive deformations

$$d\sigma_{ij}\,d\varepsilon_{ij} < 0 \tag{1.10}$$

we have the important stability condition

$$\int\limits_{\Omega} C_{ijmn}(\varepsilon, \chi = 1)\delta\varepsilon_{mn}\delta\varepsilon_{ij}d\Omega + \int\limits_{\Sigma} R_{ij}\delta u_j \delta u_i d\Sigma > 0 \tag{1.11}$$

This last inequality follows from Drucker's postulate [6] applied to the deforming and loading systems combined.

Theorem 1.1. Suppose the bounded surface Σ of a body Ω which contains the region $\Omega_0 \subset \Omega$ ($\Sigma \not\in \Omega_0$) is such that the following inequalities are satisfied

$$\Omega - \Omega_0: \quad C_{ijmn}(\varepsilon, \chi = 1)h_{mn}h_{ij} > 0; \quad \Omega_0: \quad C_{ijmn}(\varepsilon, \chi = 1)h_{mn}h_{ij} < 0 \tag{1.12}$$

where $\mathbf{h}$ is an arbitrary symmetric second-rank tensor, and the set where $C_{ijmn}(\varepsilon, \chi = 1)h_{mn}h_{ij} = 0$ has zero measure. Then (1.11) is a sufficient condition for problem (1.1), (1.3)–(1.5) to have no more than one solution.

Proof. Suppose, on the contrary, that there are two different solutions $du_i^{(1)}$, $d\varepsilon_{ij}^{(1)}$, $d\sigma_{ij}^{(1)}$ and $du_i^{(2)}$, $d\varepsilon_{ij}^{(2)}$, $d\sigma_{ij}^{(2)}$. In that case the fields

$$du_i' = du_i^{(1)} - du_i^{(2)}, \quad d\varepsilon_{ij}' = d\varepsilon_{ij}^{(1)} - d\varepsilon_{ij}^{(2)}, \quad d\sigma_{ij}' = d\sigma_{ij}^{(1)} - d\sigma_{ij}^{(2)}$$

also satisfy all the equations of the boundary-value problem with zero mass forces and boundary conditions

$$(d\sigma_{ij}'n_j + R_{ij}u_j')\,|_{\Sigma_S} = 0, \quad (du_i' + Q_{ij}d\sigma_{jk}'n_k)\,|_{\Sigma_u} = 0 \tag{1.13}$$

As we have mentioned, the boundary conditions can be reduced to the same form. In this case Eq. (1.8) takes the form

$$\int_{\Omega} d\sigma'_{ij} d\varepsilon'_{ij} d\Omega = -\int_{\Sigma=\Sigma_S} R_{ij} du'_j du'_i d\Sigma \tag{1.14}$$

The right-hand side of the last equation obviously cannot be positive. If the solution of the original boundary-value problem is not unique, the integral over the volume must be negative; otherwise both integrals are zero.

That the integrand of the volume integral is non-negative for the region $\Omega - \Omega_0$ during active loading or unloading for both solutions follows from (1.12). If just one of the solutions, the first, say, has elastic unloading in the given region, putting $\sigma^*_{ij} = \sigma_{ij} + d\sigma^{(1)}_{ij}$, from Mises maximum principle (1.2) we obtain $d\sigma^{(1)}_{ij}\, d\varepsilon^{(2)p}_{ij} \leqslant 0$. Thus in this case also the volume integral in (1.14) is non-negative. Since the right-hand side of (1.4) is non-positive, this proves that for an elastic–plastic strengthened body ($\Omega_0 = 0$) with boundary conditions in the form (1.5) the solution of the boundary-value problem is unique.

If, according to the different solutions of the boundary-value problem at each point of the region Ω_0, active loading takes place ($\chi = 1$), Eq. (1.14) cannot be satisfied if the condition for stable sub-critical deformation (1.11) holds, contradicting the initial assumption.

However, there is another possibility, in which one of the solutions, the first, say, gives elastic unloading in some region $\Omega'_0 \subseteq \Omega_0$. Bearing in mind that in that case

$$d\sigma^{(1)}_{ij} = C^e_{ijmn} d\varepsilon^{(1)}_{mn}, \quad d\sigma^{(2)}_{ij} = C^e_{ijmn} d\varepsilon^{(2)e}_{mn}$$

for any point of that region we will write

$$d\sigma'_{ij} d\varepsilon'_{ij} = C^e_{ijmn} d\varepsilon^{(1)}_{mn} d\varepsilon^{(1)}_{ij} - 2 d\sigma^{(2)}_{ij} d\varepsilon^{(1)}_{ij} - C^e_{ijmn} d\varepsilon^{(1)}_{mn} d\varepsilon^{(2)p}_{ij} + d\sigma^{(2)}_{ij} d\varepsilon^{(2)}_{ij}$$

Hence

$$d\sigma'_{ij} d\varepsilon'_{ij} - C_{ijmn}(\varepsilon, \chi = 1) d\varepsilon'_{mn} d\varepsilon'_{ij} = [C^e_{ijmn} - C_{ijmn}(\varepsilon, \chi = 1)] d\varepsilon^{(1)}_{mn} d\varepsilon^{(1)}_{ij} - d\sigma^{(1)}_{ij} d\varepsilon^{(2)p}_{ij} > 0$$

The sign of the last inequality is determined by condition (1.12) for Ω_0 and the directionality of the vectors $\mathbf{d\sigma}^{(1)}$ and $\mathbf{d\varepsilon}^{(2)p}$ into the loading surface and along its outward normal, respectively.

Thus, referring to (1.14), we can write

$$\int_{\Omega} C_{ijmn}(\varepsilon, \chi = 1) d\varepsilon'_{mn} d\varepsilon'_{ij} d\Omega + \int_{\Sigma} R_{ij} du'_i du'_j d\Sigma < 0$$

which contradicts condition (1.11) in this case also. This proves the theorem.

2. We now consider a deformed body Ω with boundary Σ and pick out from it a fictitious doubly-connected bounded region of elastic material Ω' with a rigidly fixed external boundary and an internal surface Σ', which is not very different from Σ. At each point $\mathbf{x} \in \Omega'$ Eqs (1.3), (1.4) and (1.1) hold with some constant modulus-of-elasticity tensor. We apply to the points of the surface Σ' forces dS°_i such that the displacements du°_i of the boundary points that they cause ensure the configurations of surfaces Σ' and Σ to be the same. The relation between these values is given by the equations [7]

$$du^\circ_i(\mathbf{x}') = \int_{\Sigma} G_{ij}(\mathbf{x}', \mathbf{x})\, dS^\circ_j(\mathbf{x}) d\Sigma \tag{2.1}$$

where $\mathbf{G}(\mathbf{x}', \mathbf{x})$ is Green's tensor for the region Ω'. We can also write the inverse relations

$$dS^\circ_i(\mathbf{x}') = \int_{\Sigma} N_{ij}(\mathbf{x}', \mathbf{x})\, du^\circ_j(\mathbf{x}) d\Sigma \tag{2.2}$$

The tensor $\mathbf{N}(\mathbf{x}', \mathbf{x})$, like Green's tensor, is uniquely defined by the elastic properties and geometry of the body Ω'.

We now imagine the body Ω to be placed inside the region Ω' without being deformed. Attaching the bodies perfectly along their common boundary, we remove the forces dS°_i. The combined deformation of the two regions on the boundary $\Sigma \in \Omega$ gives rise to forces

$$dS_i(\mathbf{x}') = dS_i^\circ(\mathbf{x}') - \int_\Sigma N_{ij}(\mathbf{x}', \mathbf{x})\, du_j(\mathbf{x}) d\Sigma \tag{2.3}$$

and the points on the boundary between the bodies undergo displacements

$$du_i(\mathbf{x}') = du_i^\circ(\mathbf{x}') - \int_\Sigma G_{ij}(\mathbf{x}', \mathbf{x})\, dS_j(\mathbf{x}) d\Sigma \tag{2.4}$$

If we write relation between the quantities dS_i° and du_i° in linear-tensor form (1.7), we can find the corresponding coefficients of proportionality $R_{ij}(\mathbf{x}, \mathbf{u})$ and $Q_{ij}(\mathbf{x}, \mathbf{S})$ from (2.1) and (2.2) using (1.7).

Consider a point $\mathbf{x}$ of the boundary. On the assumption that the quantities $(dS_i^\circ - dS_i)$ are independent of the quantities $(du_i^\circ - du_i)$ at every point of the boundary apart from that point, and that the variation of the stiffness coefficients $R_{ij}(\mathbf{x}, \mathbf{u})$ in the interval between $\mathbf{du}^\circ$ and $\mathbf{du}$ is negligibly small, for all $\mathbf{x} \in \Sigma$ Eqs (2.3) can be written in the simpler form

$$dS_i^\circ(\mathbf{x}) - dS_i(\mathbf{x}) = R_{ij}(\mathbf{x}, \mathbf{du}^\circ) du_j(\mathbf{x}) \tag{2.5}$$

After similar simplifying hypotheses, from (2.4), (2.1) and (1.7) we obtain the equations

$$du_i^\circ(\mathbf{x}) - du_i(\mathbf{x}) = Q_{ij}(\mathbf{x}, \mathbf{dS}^\circ) dS_j(\mathbf{x}) \tag{2.6}$$

These, like the previous ones, are essentially the same as boundary conditions (1.5). Thus, if the internal boundary Σ' of region Ω' in the undeformed state differs from Σ at each point by the corresponding displacement vector $\mathbf{du}^\circ(\mathbf{x})$, taken as the nominal displacement increment for points of the surface Σ, and the region Ω' in such that, in order for the boundaries to coincide, forces applied at points of Σ' must differ from the nominal $\mathbf{dS}^\circ(\mathbf{x})$ in sign only, then as a result of the procedure used to join Ω to the fictitious region Ω' on their common boundary, conditions of type (1.5) are established and Ω' can be used as a model of the loading system.

The fields $\mathbf{du}(\mathbf{x})$ and $\mathbf{d}\boldsymbol{\varepsilon}(\mathbf{x})$ in $\mathbf{d}\boldsymbol{\sigma}(\mathbf{x})$, caused by the fact that the original boundaries of regions Ω and Ω' are different, satisfy Eqs (1.1), (1.3) and (1.4) under conditions (1.5). We will assume that there is a loss-of-strength zone Ω_0 and that conditions (1.11) and (1.12) hold. Fields which satisfy all the given equations and inequalities will be called real.

Let $d\sigma_{ij}^*$ be statically possible stress increments in the region Ω which satisfy the equilibrium equations (1.4) and the static matching conditions

$$dS_i^*\big|_\Sigma \equiv d\sigma_{ij}^* n_j\big|_\Sigma = -dS_i^*\big|_{\Sigma'} \tag{2.7}$$

but for which, according to the constitutive relations (1.1), the corresponding possible strain increments $d\varepsilon_{ij}^*$ cannot necessarily be expressed in terms of continuous displacements. The Cauchy relations are satisfied in the region Ω', while there is a discrepancy between the statically possible fields and the real fields, due to the difference between the possible and real forces and displacements on the common boundary.

Theorem 2.1. The absolute minimum of the functional

$$W^* = \int_\Omega d\sigma_{ij}^* d\varepsilon_{ij}^* d\Omega - \int_\Sigma Q_{ij}\left(2dS_j^\circ dS_i^* - dS_j^* dS_i^* - \frac{1}{2} dS_j^\circ dS_i^\circ\right) d\Sigma \tag{2.8}$$

defined for all statically possible fields corresponds to the real field of stress increments.

Proof. Consider the following equations

$$\int_{\Omega+\Omega'} (d\sigma_{ij}^* - d\sigma_{ij}) d\varepsilon_{ij} d\Omega = \int_{\Sigma(\Omega)} (dS_i^* - dS_i) du_i d\Sigma +$$

$$+ \int_{\Sigma(\Omega')} (dS_i^* - dS_i)(du_i^\circ - du_i) d\Sigma = \int_\Sigma (dS_i^* - dS_i) du_i^\circ d\Sigma \tag{2.9}$$

obtained by applying the Cauchy relations to the actual deformation field, and using the equilibrium equations and the Gauss–Ostrogradskii theorem.

We can represent the integrand on the left-hand side of (2.9) in the form

$$(d\sigma_{ij}^* - d\sigma_{ij})d\varepsilon_{ij} = \frac{1}{2}(d\sigma_{ij}^* d\varepsilon_{ij}^* - d\sigma_{ij} d\varepsilon_{ij}) - \frac{1}{2}(d\sigma_{ij}^* - d\sigma_{ij})(d\varepsilon_{ij}^* - d\varepsilon_{ij}) \tag{2.10}$$

Allowing for possible unloading, we have

$$\int_\Omega (d\sigma_{ij}^* - d\sigma_{ij})(d\varepsilon_{ij}^* - d\varepsilon_{ij})d\Omega \geqslant \int_\Omega C_{ijmn}(\varepsilon, \chi = 1)[d\varepsilon_{mn}^* - d\varepsilon_{mn}][d\varepsilon_{ij}^* - d\varepsilon_{ij}]d\Omega \tag{2.11}$$

which holds by virtue of the fact that the material possesses a soft characteristic and follows from the analogous inequality obtained for the region Ω_0 in the proof of Theorem 1.1.

Then

$$\int_{\Omega'} (d\sigma_{ij}^* - d\sigma_{ij})(d\varepsilon_{ij}^* - d\varepsilon_{ij})d\Omega =$$

$$= \int_\Sigma (dS_i^* - dS_i)[(du_i^\circ - du_i^*) - (du_i^\circ - du_i)]d\Sigma = \int_\Sigma R_{ij}(du_j^* - du_j)(du_i^* - du_i)d\Sigma \tag{2.12}$$

Returning to (2.9), from (2.10)–(2.12) and condition (1.11) we obtain

$$\int_{\Omega+\Omega'} (d\sigma_{ij}^* d\varepsilon_{ij}^* - d\sigma_{ij} d\varepsilon_{ij})d\Omega \geqslant 2\int_\Sigma (dS_i^* - dS_i)du_i^\circ d\Sigma$$

The equality holds when the statically possible and actual fields are the same.

According to the equation of virtual work and the conjugacy conditions (2.7) and (2.6)

$$\int_{\Omega'} d\sigma_{ij}^* d\varepsilon_{ij}^* d\Omega = \int_\Sigma dS_i^*(du_i^\circ - du_i^*)d\Sigma = \int_\Sigma Q_{ij} dS_j^* dS_i^* d\Sigma$$

$$\int_{\Omega'} d\sigma_{ij} d\varepsilon_{ij} d\Omega = \int_\Sigma dS_i(du_i^\circ - du_i)d\Sigma = \int_\Sigma Q_{ij} dS_j dS_i d\Sigma$$

Hence

$$\int_\Omega d\sigma_{ij}^* d\varepsilon_{ij}^* d\Omega - \int_\Sigma (2dS_i^* du_i^\circ - Q_{ij} dS_j^* dS_i^*)d\Sigma \geqslant$$

$$\geqslant \int_\Omega d\sigma_{ij} d\varepsilon_{ij} d\Omega - \int_\Sigma (2dS_i du_i^\circ - Q_{ij} dS_j dS_i)d\Sigma = \int_\Sigma dS_i du_i^\circ d\Sigma \tag{2.13}$$

This proves the given extremum principle.

In the special case where conditions of rigid loading ($Q_{ij} = 0$) are given on Σ_u, part of the surface Σ, and conditions of soft loading ($R_{ij} = 0$) on Σ_S, another part of the surface, the statically possible fields must satisfy the equation

$$d\sigma_{ij}^* n_j \big|_{\Sigma_S} = dS_i^\circ$$

and relation (2.13) takes the form

$$\frac{1}{2}\int_\Omega d\sigma_{ij}^* d\varepsilon_{ij}^* d\Omega - \int_{\Sigma_u} dS_i^* du_i d\Sigma \geqslant \frac{1}{2}\int_\Omega d\sigma_{ij} d\varepsilon_{ij} d\Omega - \int_{\Sigma_u} dS_i du_i d\Sigma =$$

$$= \frac{1}{2}\int_{\Sigma_S} dS_i du_i d\Sigma - \frac{1}{2}\int_{\Sigma_u} dS_i du_i d\Sigma$$

and is the same as the expression of the extremum principle obtained using traditional boundary conditions [8].

For statically permissible fields which differ infinitesimally from the actual field ($d\sigma_{ij}^* = d\sigma_{ij} + \delta(d\sigma_{ij})$), the functional W^* takes an extremal value, provided that it is stationary with respect to variations $\delta(d\sigma_{ij})$ which satisfy the equilibrium equations. In that case the equation

$$\int_\Omega \delta(d\sigma_{ij})d\varepsilon_{ij}d\Omega - \int_\Sigma \delta(dS_i)[du_i^\circ - Q_{ij}dS_j]d\Sigma = 0$$

expresses the modified variational principle for elastic–plastic bodies with possible loss-of-strength zones and boundary conditions of the contact type.

3. The second extremum principle concerns the kinematically possible strain increments $d\bar{\varepsilon}_{ij}$, which are associated with the displacement increments $d\bar{u}_i$ by Cauchy's relations and, on the boundary of regions Ω and Ω', satisfy the kinematic conjugacy conditions

$$d\bar{u}_i \,|_\Sigma = d\bar{u}_i \,|_{\Sigma'} \tag{3.1}$$

but for which the possible stress increments $d\bar{\sigma}_{ij}$, corresponding to the constitutive relations, do not necessarily satisfy the equilibrium equations in region Ω.

Theorem 3.1. The absolute maximum of the functional

$$\overline{W} = \int_\Sigma R_{ij}\left(2du_j^\circ d\bar{u}_i - d\bar{u}_j d\bar{u}_i - \frac{1}{2}du_j^\circ du_i^\circ\right)d\Sigma - \int_\Omega d\bar{\sigma}_{ij}d\bar{\varepsilon}_{ij}d\Omega \tag{3.2}$$

defined for all kinematically possible fields corresponds to the actual field of strain increments.

Proof. Consider the integral

$$\int_{\Omega+\Omega'} (d\bar{\varepsilon}_{ij} - d\varepsilon_{ij})d\sigma_{ij}d\Omega = \int_{\Sigma(\Omega)} (d\bar{u}_i - du_i)dS_i d\Sigma +$$

$$+ \int_{\Sigma(\Omega')} [(du_i^\circ - d\bar{u}_i) - (du_i^\circ - du_i)]dS_i d\Sigma = 0 \tag{3.3}$$

and the identity

$$2(d\bar{\varepsilon}_{ij} - d\varepsilon_{ij})d\sigma_{ij} \equiv f_1 - f_2$$

$$f_1 = d\bar{\sigma}_{ij}d\bar{\varepsilon}_{ij} - d\sigma_{ij}d\varepsilon_{ij}, \quad f_2 = d\bar{\varepsilon}_{ij}(d\bar{\sigma}_{ij} - d\sigma_{ij}) + d\sigma_{ij}(d\varepsilon_{ij} - d\bar{\varepsilon}_{ij})$$

We will determine the sign of the following quantity

$$\int_{\Omega+\Omega'} A d\Omega = \int_{\Omega+\Omega'} (f_2 - f_3)d\Omega, \quad f_3 = C_{ijmn}(\varepsilon, \chi = 1)[d\bar{\varepsilon}_{mn} - d\varepsilon_{mn}][d\bar{\varepsilon}_{ij} - d\varepsilon_{ij}]$$

In regions of active loading for all kinematically possible and actual continuations of the process $A = 0$. In zones of elastic strain and unloading, both for $d\bar{\sigma}_{ij}$ and for $d\sigma_{ij}$, we have

$$A = [C_{ijmn}^e - C_{ijmn}(\varepsilon, \chi = 1)](d\bar{\varepsilon}_{mn} - d\varepsilon_{mn})(d\bar{\varepsilon}_{ij} - d\varepsilon_{ij}) \geqslant 0$$

which is governed by the properties of the materials mentioned above. We arrive at a similar expression for A when $d\sigma_{ij}$ produce the loading and $d\bar{\sigma}_{ij}$ produce the unloading.

If active loading corresponds to the kinematically possible strain increment $d\bar{\varepsilon}_{ij}$, and elastic unloading corresponds to the actual increments $d\varepsilon_{ij}$, then

$$A = [C_{ijmn}^e - C_{ijmn}(\varepsilon, \chi = 1)]d\varepsilon_{mn}d\varepsilon_{ij} - 2d\sigma_{ij}d\bar{\varepsilon}_{ij}^p > 0$$

The validity of a similar inequality has already been demonstrated in the proof of Theorem 1.1.

From the equation of virtual work for the region Ω' and condition (1.11) with $d\bar{\varepsilon}_{ij} \neq d\varepsilon_{ij}$, we obtain

$$\int_{\Omega+\Omega'} f_3 d\Omega = \int_\Omega f_3 d\Omega + \int_\Sigma R_{ij}(d\bar{u}_j - du_j)(d\bar{u}_i - du_i)d\Sigma > 0$$

Thus we have proved that

$$\int_{\Omega+\Omega'} [d\bar{\varepsilon}_{ij}(d\bar{\sigma}_{ij} - d\sigma_{ij}) + d\sigma_{ij}(d\varepsilon_{ij} - d\bar{\varepsilon}_{ij})]d\Omega \geqslant 0$$

and therefore

$$\frac{1}{2}\int_{\Omega+\Omega'}(d\bar{\sigma}_{ij}d\bar{\varepsilon}_{ij}-d\sigma_{ij}d\varepsilon_{ij})d\Omega \geqslant \int_{\Omega+\Omega'}(d\bar{\varepsilon}_{ij}-d\varepsilon_{ij})\,d\sigma_{ij}d\Omega \tag{3.4}$$

According to the conjugacy conditions (3.1) and Eqs (2.5), we have

$$\int_{\Omega'} d\bar{\sigma}_{ij}d\bar{\varepsilon}_{ij}d\Omega = \int_{\Sigma}(dS_i^\circ - R_{ij}d\bar{u}_j)(du_i^\circ - d\bar{u}_i)d\Sigma =$$

$$= \int_{\Sigma}(dS_i^\circ du_i^\circ - 2dS_i^\circ d\bar{u}_i + R_{ij}d\bar{u}_j d\bar{u}_i)d\Sigma$$

and the analogous equation with $\bar{\sigma}_{ij}$, $\bar{\varepsilon}_{ij}$ and $\bar{u}_i$ replaced by σ_{ij}, ε_{ij} and u_i.

Returning to inequality (3.4), from (3.3) and the last relations we obtain

$$\int_{\Omega} d\bar{\sigma}_{ij}d\bar{\varepsilon}_{ij}d\Omega - \int_{\Sigma}(2dS_i^\circ d\bar{u}_i - R_{ij}d\bar{u}_j d\bar{u}_i)d\Sigma \geqslant$$

$$\geqslant \int_{\Omega} d\sigma_{ij}d\varepsilon_{ij}d\Omega - \int_{\Sigma}(2dS_i^\circ du_i - R_{ij}du_j du_i)d\Sigma = -\int_{\Sigma} dS_i^\circ du_i d\Sigma \tag{3.5}$$

This proves the extremum principle.

In the special case where

$$R_{ij}\big|_{\Sigma_S} = 0, \quad Q_{ij}\big|_{\Sigma_u} = 0, \quad \bar{u}_i\big|_{\Sigma_u} = u_i\big|_{\Sigma_u} = u_i^\circ$$

inequality (3.5) has the form

$$\int_{\Sigma_S} dS_i d\bar{u}_i d\Sigma - \frac{1}{2}\int_{\Omega} d\bar{\sigma}_{ij}d\bar{\varepsilon}_{ij}d\Omega \leqslant$$

$$\leqslant \int_{\Sigma_S} dS_i du_i d\Sigma - \frac{1}{2}\int_{\Omega} d\sigma_{ij}d\varepsilon_{ij}d\Omega = \frac{1}{2}\int_{\Sigma_S} dS_i du_i d\Sigma - \frac{1}{2}\int_{\Sigma_u} dS_i du_i d\Sigma$$

and is the same as the expression of the well-known extremum principle obtained using traditional boundary conditions [8].

For kinematically admissible fields which differ infinitesimally from the actual field ($d\bar{\varepsilon}_{ij} = d\varepsilon_{ij} + \delta(d\varepsilon_{ij})$), the functional $\bar{W}$ takes an extremal value provided that it is stationary with respect to variations $\delta(d\varepsilon_{ij})$ which satisfy the Cauchy relations. In that case, the equation

$$\int_{\Omega} d\sigma_{ij}\delta(d\varepsilon_{ij})d\Omega - \int_{\Sigma}\delta(du_i)[dS_i^\circ - R_{ij}du_j]d\Sigma = 0$$

expresses the second modified variational principle for elastic–plastic bodies with possible loss-of-strength zones and boundary conditions of the contact type.

According to the principles thus formulated

$$W^* \geqslant W \geqslant \bar{W}$$

where

$$W = \int_{\Sigma}\left(\frac{1}{2}dS_i^\circ - dS_i\right)du_i^\circ d\Sigma = \int_{\Sigma}\left(du_i - \frac{1}{2}du_i^\circ\right)dS_i^\circ d\Sigma$$

and we therefore have the conditions for obtaining the upper and lower bounds in the approximate solution of the boundary-value problems.

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